On the Resonant Frequency of a Drum

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Introduction

The equations for a vibrating circular membrane give normal frequencies which are not in a harmonic relationship. Traditionally, this result is interpreted as the reason that drums are used for rhythm rather than melody.[1] The problem here is that certain modern drums are used for rhythm *and* melody. If identifiable pitches are being perceived in these drums, then how are the necessary harmonic frequencies being produced? This question suggests the development of a more detailed mathematical model - one which treats the drum as a system of several vibrating components.

The orchestral tympani, or kettle-drum, consists of a single membrane stretched over a bowl-shaped shell. It produces powerful bass notes with a definite sense of pitch, and Lord Rayleigh identified at least a partial sequence of harmonic frequencies emanating from the anti-symmetric modes of the membrane.[2] These observations have been very nearly explained with a model that accounts for the tympani's enclosed volume of air. This approach supposes that the air essentially adds mass to the membrane, and Green's functions are used to approximate the air loading.[3] This model produces a shifted set of modal frequencies which closely agree with the data obtained from an actual drum. However, the more puzzling phenomenon of resonance has been observed in two-headed drums like tom-toms and bass drums.

The tonal bass drums used in marching percussion are large, loud drums of various diameters which are carried vertically and can be struck from either side. As a consequence, both membranes are the same thickness, and usually have the same the same tension. These drums are tuned to harmonic intervals, and some percussionists claim that each bass can be identified with a definite pitch. Furthermore, it has been observed that for each drum and head combination, there is an optimum tuning scheme - one which will produce the clearest and most lasting tone when the drum is struck.[4] Similar in design are Japanese Taiko drums which consist of two membranes stretched over a long, barrel-shaped wooden shell. These drums were studied in 1934 by Obata and Tesima, and the coupled vibration of the top and bottom heads is beautifully shown in Figure 1(a).[5]

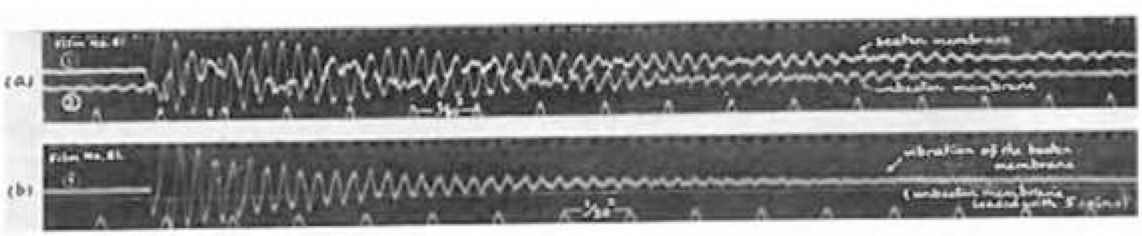

Figure 1: (a) Vibrations of top and bottom membranes in a Taiko Drum. (b) Taiko Drum with damped bottom membrane. J. Obata and T. Tesima, "Experimental studies on the sound and vibration of drums," Journal of the Acoustical Society of America 6, pp 267 (1935)

In light of this data, viewing the air as "mass-loading" for the membranes would not account for the transfer of energy between the two heads, and indeed that model's predicted frequencies do not agree with the experimentally determined frequency spectrum of a bass drum.[3] In fact, the patterns of head displacements recorded experimentally suggest the "beats" produced by superposition of the modes of a coupled system of two masses, the coupling in this case being relatively weak. This would imply that the enclosed air is acting more like a spring than a dashpot.

In this paper, we develop and study a simplified model of a tympani-style drum, treating the air as a spring. These ideas are then used to model a two-headed drum as a coupled system of vibrating membranes.

One Membrane

The Model

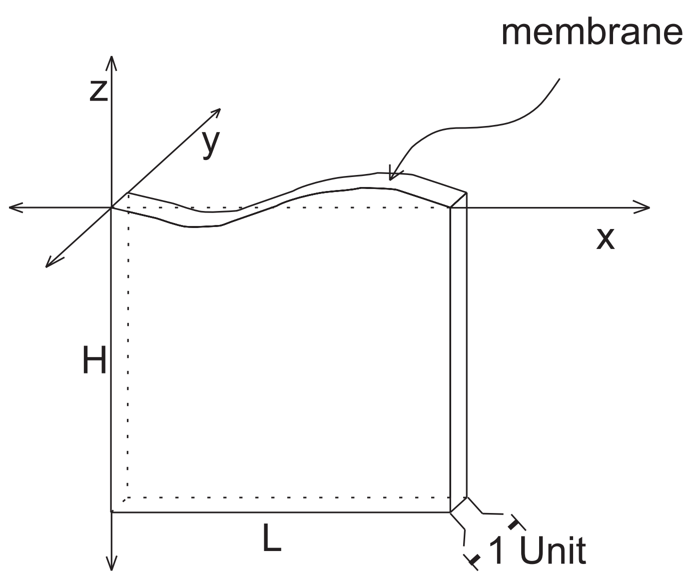

Figure 2: Geometry of the idealized drum with a single membrane.

The simplest model that still exhibits the desired characteristics is a flexible elastic membrane stretched over a roughly rectangular enclosure of air with rigid sides of height 'H', a rigid bottom of length 'L' and a width of one unit. If the membrane is allowed only vertical displacements, with no twisting or side-to-side movement, then the membrane's behavior over time can be modelled by the function

$$u(t,x), \quad 0 < x < L, \quad 0 < t$$

where the displacement is independent of y. Since we expect the displacements to be relatively small, we assume the membrane is held at a constant tension τ and has constant density ρ^m. Now in order to describe the function $u(t,x)$, we consider a small section of the membrane starting at a generic point $x \epsilon [0,L]$ and ending at $x + \Delta x$, and examine the forces acting on it (Figure 3).

In order to define the upward force f^a due to the compression of air inside the box we first note that when the membrane is at rest, *i.e.* $u(t,x) = 0 \quad \forall x \epsilon [0,L]$, the atmospheric pressure inside and outside the box will apply equal and opposite forces to the respective surfaces of

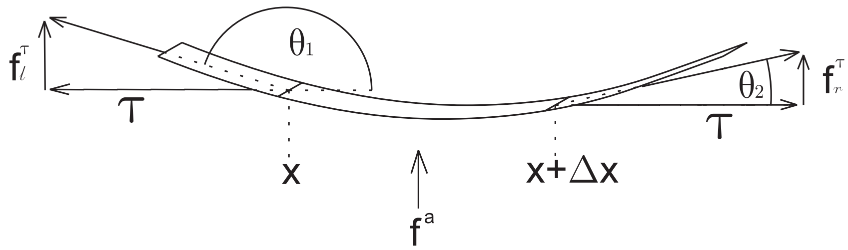

Figure 3: Forces acting on a small section of the membrane

the membrane. Here we assume that air pressure is proportional to air density. This idea goes back to Helmholz[6] , and was described in the book by Lamb[7] in reference to resonators. Thus the atmospheric pressure is $P_0^a = \gamma\rho_0^a$, where γ is some proportionality constant, and ρ_0^a is the density of air under standard conditions. Now whenever the volume of air inside the box changes, the density and hence the pressure of the air inside the box changes, so the box's air's pressure as a function of the excess volume Δv is given by

$$P(\Delta v) = \gamma\rho(\Delta v) = \gamma\left(\frac{\rho_0^a HL}{HL + \Delta v}\right).$$

Here we are only concerned with the *excess* pressure

$$P_x(\Delta v) = P(\Delta v) - P_0^a = \frac{\gamma\rho_0^a HL}{HL + \Delta v} - \gamma\rho_0^a$$

$$\text{or} \qquad P_x(\Delta v) = \frac{-\gamma\rho_0^a \Delta v}{HL + \Delta v}.$$

Assuming that Δv will be relatively small compared to the total volume HL, we approximate $P_x(\Delta v)$ around $\Delta v = 0$ with a Taylor polynomial, keeping only the first-order term:

$$\Delta P_x(\Delta v) = \frac{-\gamma\rho_0^a \Delta v}{HL}.$$

Thus the upward force due to air pressure on the section of length Δx is given by

$$\Delta x \Delta P_x(\Delta v) = \frac{-\Delta x \gamma\rho_0^a \Delta v}{HL}.$$

As for the upward forces due to tension, we see from Figure 3 that

$$\frac{f_l^\tau}{-\tau} = \tan(\theta_1), \quad \text{and} \quad \frac{f_r^\tau}{\tau} = \tan(\theta_2)$$

$$\text{or} \qquad f_l^\tau = -\tau\left(\frac{\partial u(t,x)}{\partial x}\right), \quad \text{and} \quad f_r^\tau = \tau\frac{\partial u(t,x+\Delta x)}{\partial x}$$

so the sum of forces acting on the section of membrane is given by

$$f = \tau\left[\frac{\partial u(t,x+\Delta x)}{\partial x} - \frac{\partial u(t,x)}{\partial x}\right] - \frac{\Delta x \gamma\rho_0^a \Delta v}{HL}.$$

Thus from Newton's second law, $ma = f$, we can write

$$\Delta x \rho^m \frac{\partial^2 u(t,x)}{\partial t^2} = \tau\left[\frac{\partial u(t,x+\Delta x)}{\partial x} - \frac{\partial u(t,x)}{\partial x}\right] - \frac{\Delta x \gamma\rho_0^a \Delta v}{HL}.$$

Dividing both sides by Δx and taking the limit as $\Delta x \to 0$, we obtain

$$\rho^m \frac{\partial^2 u(t,x)}{\partial t^2} = \tau \frac{\partial^2 u(t,x)}{\partial x^2} - \frac{\gamma \rho_0^a \Delta v}{HL}.$$

In this equation, the excess volume is given by

$$\Delta v = \int_0^L u(t,x)dx,$$

and so upon substituting into the above equation and dividing both sides by ρ^m, we obtain the integro-differential equation

$$(1) \qquad \frac{\partial^2 u(t,x)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(t,x)}{\partial x^2} - C \int_0^L u(t,x)dx$$

$$where \qquad \alpha^2 = \frac{\tau}{\rho^m}, \quad and \quad C = \frac{\gamma \rho_0^a}{HL\rho^m},$$

along with the boundary conditions

$$(2) \qquad u(t,0) = 0 = u(t,L), \quad t > 0$$

which specify that the membrane is fixed at the ends $x = 0, x = L$.

In trying to find solutions to (1) we employ the method of separation of variables, assuming $u(t,x) = T(t)X(x)$. Then (1) becomes

$$T''(t)X(x) = \alpha^2 T(t)X''(x) - CT(t) \int_0^L X(x)dx.$$

Dividing both sides by $\alpha^2 T(t)X(x)$ gives

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} - \frac{C \int_0^L X(x)dx}{\alpha^2 X(x)}.$$

Since t and x are supposed to be independent, the left and right sides of the above expression should be constant, i.e.

$$\frac{T''(t)}{\alpha^2 T(t)} = \Lambda = \frac{X''(x)}{X(x)} - \frac{C \int_0^L X(x)dx}{\alpha^2 X(x)}$$

for some constant Λ.

$$Thus, \qquad (3) \qquad T''(t) - \Lambda\alpha^2 T(t) = 0$$

$$and \qquad (4) \qquad \Lambda X(x) = X''(x) - \frac{C}{\alpha^2} \int_0^L X(x)dx$$

where the boundary conditions in (2) imply that

$$(5) \qquad X(0) = 0 = X(L).$$

Now we are in a position to state the following.

Theorem:

The integro-differential equation (4) with boundary conditions (5) has the set of eigenfunctions

$$S = \left\{ \cos\left(\frac{2v_n\pi x}{L} - v_n\pi\right) - \cos(v_n\pi) \right\}_{n=1}^{\infty} \bigcup \left\{ \sin\left(\frac{2m\pi x}{L}\right) \right\}_{m=1}^{\infty}$$

where each v_n solves

$$(6) \qquad \tan(\pi v) = \pi v - \frac{4\alpha^2\pi^3 v^3}{CL^3}$$

with $v_n \to \frac{2n-1}{2}$ as $n \to \infty$. Moreover, the set S is orthogonal, and complete with respect to the space $L^2[0, L]$.

Proof:

We start with equation (4). In trying to find solutions, the main intution is this: the mode shapes involved in the solutions to (1) should be similar to the mode shapes of the well known vibrating string problem, except they will be “squashed” or compressed due the the effect of the air enclosed beneath them. With this in mind, we look for solutions of the form

$$X(x) = \sin(\pi\omega x + \phi) + D.$$

Since $X(0) = 0$, we have

$$\sin(\phi) + D = 0 \qquad \text{or} \quad D = -\sin(\phi),$$

and since $X(L) = 0$,

$$\sin(\pi\omega L + \phi) - \sin(\phi) = 0$$

$$\text{or} \quad \sin(\pi\omega L + \phi) = \sin(\phi).$$

The above equation gives two cases:

$$(i) \qquad \pi\omega L = 2m\pi, \quad m = 1, 2, \ldots, \quad \text{or}$$

$$(ii) \qquad \pi\omega L + \phi = \pi - \phi + 2m\pi, \quad m = 1, 2, \ldots .$$

Under case (i) we have

$$X(x) = \sin(\frac{2m\pi x}{L} + \phi) - \sin(\phi),$$

$$X''(x) = \frac{-4m^2\pi^2}{L^2}\sin(\frac{2m\pi x}{L} + \phi),$$

$$\text{and} \quad \int_0^L X(x)dx = \left\{ \frac{-L}{2m\pi}\cos(\frac{2m\pi x}{L} + \phi) - x\sin(\phi) \right]_0^L = -L\sin(\phi).$$

Thus (4) becomes

$$\Lambda\left[\sin(\frac{2m\pi x}{L} + \phi) - \sin(\phi)\right] = \frac{-4m^2\pi^2}{L^2}\sin(\frac{2m\pi x}{L} + \phi) + \frac{CL}{\alpha^2}\sin(\phi)$$

from which we see that $\Lambda = \frac{-4m^2\pi^2}{L^2}$, and

$$\begin{aligned} -\Lambda \sin(\phi) &= \frac{CL}{\alpha^2}\sin(\phi) \\ \textit{or} \quad \left[\frac{CL}{\alpha^2} - \frac{4m^2\pi^2}{L^2}\right]\sin(\phi) &= 0, \\ \textit{i.e.} \quad \left[\frac{CL^3 - 4m^2\pi^2\alpha^2}{\alpha^2 L^2}\right]\sin(\phi) &= 0. \end{aligned}$$

Thus if $CL^3 - 4m^2\pi^2\alpha^2 \neq 0 \quad \forall m = 1, 2, \ldots$ [†], then $\sin(\phi) = 0$ which implies $\phi = z\pi, z = \pm 1, \pm 2, \ldots$, and

$$X(x) = \pm \sin(\frac{2m\pi x}{L}), \quad m = 1, 2, \ldots$$

is a solution to (4) with

$$\Lambda_{2m} = \frac{-4m^2\pi^2}{L^2}.$$

As for case (ii),

$$\pi\omega L + \phi = \pi - \phi + 2m\pi, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad \phi = \frac{-\pi\omega L}{2} + \frac{(2m+1)\pi}{2}, \quad m = 1, 2, \ldots$$

and

$$\begin{aligned} X(x) &= \sin\left(\pi\omega x - \frac{\pi\omega L}{2} + \frac{(2m+1)\pi}{2}\right) - \sin\left(\frac{-\pi\omega L}{2} + \frac{(2m+1)\pi}{2}\right) \\ &= \pm\left[\cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - \cos\left(\frac{-\pi\omega L}{2}\right)\right] \\ \textit{or} \quad X(x) &= \pm\left[\cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - \cos\left(\frac{\pi\omega L}{2}\right)\right]. \end{aligned}$$

Assuming without loss of generality that

$$X(x) = \cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - \cos\left(\frac{\pi\omega L}{2}\right)$$

we have

$$X''(x) = -\pi^2\omega^2 \cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right),$$

$$\begin{aligned} \textit{and} \quad \int_0^L X(x)dx &= \left\{\frac{1}{\pi\omega}\sin\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - x\cos\left(\frac{\pi\omega L}{2}\right)\right]_0^L \\ &= \frac{2}{\pi\omega}\sin\left(\frac{\pi\omega L}{2}\right) - L\cos\left(\frac{\pi\omega L}{2}\right). \end{aligned}$$

Substituting into (4) we see that

$$\Lambda\left[\cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - \cos\left(\frac{\pi\omega L}{2}\right)\right] = -\pi^2\omega^2\cos\left(\pi\omega x - \frac{\pi\omega L}{2}\right) - \frac{C}{\alpha^2}\left\{\frac{2}{\pi\omega}\sin\left(\frac{\pi\omega L}{2}\right) - L\cos\left(\frac{\pi\omega L}{2}\right)\right\}$$

[†] In the remark at the end of the proof, we discuss the special case where $CL^3 - 4(m^*)^2\pi^2\alpha^2 = 0$ for some integer m^*

$$\textit{so} \qquad \Lambda = -\pi^2\omega^2, \quad \textit{and} \quad -\Lambda\cos\left(\frac{\pi\omega L}{2}\right) = -\frac{2C}{\alpha^2\pi\omega}\sin\left(\frac{\pi\omega L}{2}\right) + \frac{CL}{\alpha^2}\cos\left(\frac{\pi\omega L}{2}\right)$$

$$\textit{or} \quad \left[\pi^2\omega^2 - \frac{CL}{\alpha^2}\right]\cos\left(\frac{\pi\omega L}{2}\right) = -\frac{2C}{\alpha^2\pi\omega}\sin\left(\frac{\pi\omega L}{2}\right).$$

If we define a new variable v by

$$v = \frac{\omega L}{2},$$

then $\omega = \frac{2v}{L}$, and from above we have

$$\frac{\frac{4\pi^2v^2}{L^2} - \frac{CL}{\alpha^2}}{\frac{-CL}{\alpha^2\pi v}} = \tan(\pi v)$$

$$\textit{or} \qquad \frac{\frac{4\alpha^2\pi^3v^3}{L^2} - CL\pi v}{-CL} = \tan(\pi v)$$

$$\textit{i.e.} \qquad \tan(\pi v) = \pi v - \frac{4\alpha^2\pi^3v^3}{CL^3}.$$

Thus

$$X_{2v_n}(x) = \cos\left(\frac{2v_n\pi x}{L} - v_n\pi\right) - \cos(v_n\pi)$$

is a solution to (4), where

$$(6) \qquad \tan(\pi v_n) = \pi v_n - \frac{4\alpha^2\pi^3v_n^3}{CL^3}, \quad n = 1, 2, \ldots$$

and $v_n \to \frac{2n-1}{2}$ as $n \to \infty$. From the plot of $\tan(\pi v)$ versus $\pi v - \frac{4\alpha^2\pi^3v^3}{CL^3}$ (with $\alpha^2 = L = 1$, and $C = 4\pi^2$) shown in Figure 4, it is clear that for each v_n, we have $\frac{2n-1}{2} < v_n < \frac{2n+1}{2}$. These

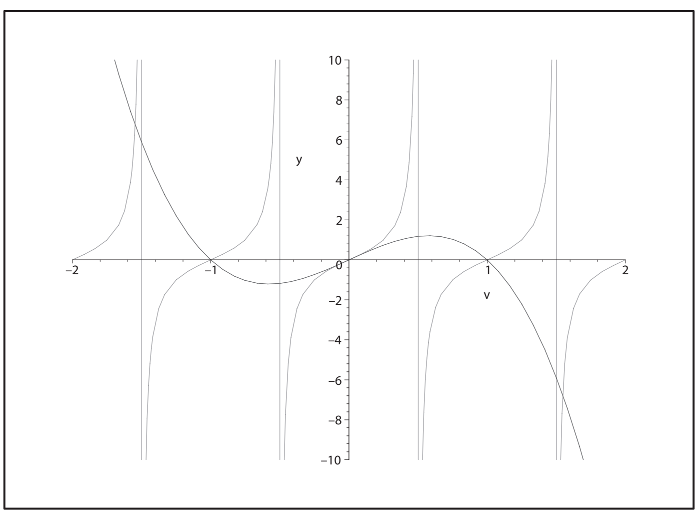

Figure 4: $\tan(\pi v)$ vs. $\pi v - \frac{4\alpha^2\pi^3v^3}{CL^3}$

solutions give $\Lambda_{2v_n} = -\frac{4v_n^2\pi^2}{L^2}$.

We now consider the orthogonality of the set of eigenfunctions defined by

$$S = \left\{ \cos\left(\frac{2v_n\pi x}{L} - v_n\pi\right) - \cos(v_n\pi) \right\}_{n=1}^{\infty} \bigcup \left\{ \sin\left(\frac{2m\pi x}{L}\right) \right\}_{m=1}^{\infty},$$

where each v_n satisfies (6). Now if $H_0^2[0,L]$ is the set of all square integrable, twice differentiable functions $X : [0,L] \to \mathbf{R}$ such that $X(0) = X(L) = 0$, then S is a set of eigenfunctions for the linear operator $F : H_0^2[0,L] \to L^2[0,L]$ defined by

$$F(X)(x) = X''(x) - \frac{C}{\alpha^2}\int_0^L X(x)dx.$$

These functions will be orthogonal if F is self-adjoint. Indeed, if we consider the inner product on $L^2[0,L]$ defined by $\langle X, Y\rangle = \int_0^L X(s)Y(s)ds$, then

$$\begin{aligned}
\langle F(X), Y\rangle &= \int_0^L \left[\left(X''(s) - \frac{C}{\alpha^2}\int_0^L X(x)dx\right)Y(s)\right] ds \\
&= \int_0^L X''(s)Y(s)ds - \frac{C}{\alpha^2}\int_0^L \left(\int_0^L X(x)dx\right)Y(s)ds \\
&= \left(\{X'(s)Y(s)]_0^L - \int_0^L X'(s)Y'(s)ds\right) - \frac{C}{\alpha^2}\int_0^L\int_0^L X(x)Y(s)dxds.
\end{aligned}$$

Since $Y \epsilon H_0^2[0,L]$, $Y(0) = 0 = Y(L) \quad \Rightarrow \{X'(s)Y(s)]_0^L = 0$ and

$$\begin{aligned}
\langle F(X), Y\rangle &= -\int_0^L X'(s)Y'(s)ds - \frac{C}{\alpha^2}\int_0^L\int_0^L X(x)Y(s)dxds \\
&= -\left[\{X(s)Y'(s)]_0^L - \int_0^L X(s)Y''(s)ds\right] - \frac{C}{\alpha^2}\int_0^L\int_0^L X(s)Y(x)dxds.
\end{aligned}$$

Similarly $\{X(s)Y'(s)]_0^L = 0$ so

$$\begin{aligned}
\langle F(X), Y\rangle &= \int_0^L X(s)Y''(s)ds - \frac{C}{\alpha^2}\int_0^L X(s)\left(\int_0^L Y(x)dx\right)ds \\
&= \int_0^L X(s)\left(Y''(s) - \frac{C}{\alpha^2}\int_0^L Y(x)dx\right)ds \\
&= \langle X, F(Y)\rangle,
\end{aligned}$$

which shows that F is self-adjoint, as desired. Thus the eigenfunctions in S are orthogonal with respect to the aforementioned inner product.

In order to show completeness of the set S we will show that the inverse of the operator F is compact. To find the inverse, we consider an arbitrary element $X \epsilon H_0^2$ and let

$$f(x) = F[X](x).$$

Thus

$$f(x) = X''(x) - \frac{C}{\alpha^2}\int_0^L X(x)dx$$

$$\textit{or} \qquad X''(x) = f(x) + D, \qquad \textit{where} \quad D = \frac{C}{\alpha^2}\int_0^1 X(x)dx.$$

Upon integrating twice, we obtain the equation

$$X(x) = \int_0^x \int_0^t f(s)dsdt + \frac{Dx^2}{2} + Ex + F,$$

where $X(0) = 0 \quad \Rightarrow \quad F = 0$. Also, by examining the region of integration, we see that

$$\int_0^x \int_0^t f(s)dsdt = \int_0^x \int_s^x f(s)dtds$$

so

$$X(x) = \int_0^x \int_s^x f(s)dtds + \frac{Dx^2}{2} + Ex,$$

or

$$X(x) = \int_0^x (x-s)f(s)ds + \frac{Dx^2}{2} + Ex.$$

Now $X(L) = 0$ implies that

$$0 = \int_0^L (L-s)f(s)ds + \frac{DL^2}{2} + EL$$

$$\textit{or} \qquad E = \frac{1}{L}\int_0^L (s-L)f(s)ds - \frac{DL}{2},$$

so we have

$$X(x) = \int_0^x (x-s)f(s)ds + \frac{Dx^2}{2} + \frac{x}{L}\int_0^L (s-L)f(s)ds - \frac{DLx}{2}.$$

We break up the second integral and combine it with the first to obtain

$$\begin{aligned} X(x) &= \int_0^x (x - s + \frac{xs}{L} - x)f(s)ds + \int_x^L \frac{x(s-L)}{L}f(s)ds + \frac{Dx^2}{2} - \frac{DLx}{2} \\ &= \int_0^x \left(\frac{x}{L} - 1\right) sf(s)ds + \int_x^L x\left(\frac{s}{L} - 1\right) f(s)ds + \frac{D}{2}(x^2 - Lx) \\ \textit{or} \qquad X(x) &= \int_0^L G(x,s)f(s)ds + \frac{D}{2}(x^2 - Lx) \end{aligned}$$

$$\textit{where} \qquad G(x,s) = \begin{cases} \left(\frac{x}{L} - 1\right)s, & 0 \le s \le x, \\ x\left(\frac{s}{L} - 1\right), & x < s \le L. \end{cases}$$

Since $D = \frac{C}{\alpha^2}\int_0^L X(x)dx$, we integrate the above expression with respect to x, obtaining

$$D = \frac{C}{\alpha^2}\int_0^L \int_0^L G(x,s)f(s)dsdx + \frac{CD}{2\alpha^2}\int_0^L (x^2 - Lx)dx$$

$$\textit{or} \qquad D = \frac{C}{\alpha^2}\int_0^L \left(\int_0^L G(x,s)dx\right) f(s)ds - \frac{CDL^3}{12\alpha^2}$$

where we have switched the order of integration in the first integral, and evaluated the second. Noting that the function G is symmetric about the line $x = s$, we see that

$$\int_0^L G(x,s)dx = \int_0^L G(s,x)dx = \frac{1}{2}\left(\frac{s}{L} - 1\right)s$$

so

$$\begin{aligned} D &= \frac{C}{\alpha^2}\int_0^L \frac{1}{2}\left(\frac{s}{L}-1\right)sf(s)ds - \frac{CDL^3}{12\alpha^2} \\ &= \frac{C}{2\alpha^2\left(1+\frac{CL^3}{12\alpha^2}\right)}\int_0^L \left(\frac{s}{L}-1\right)sf(s)ds \\ &= \frac{6C}{12\alpha^2+CL^3}\int_0^L \left(\frac{s}{L}-1\right)sf(s)ds. \end{aligned}$$

Thus

$$X(x)=\int_0^L G(x,s)f(s)ds + \frac{3C}{12\alpha^2+CL^3}\int_0^L x(x-L)\left(\frac{s}{L}-1\right)sf(s)ds$$

or

$$X(x)=\int_0^L G(x,s)f(s)ds + \int_0^L H(x,s)f(s)ds$$

$$\textit{where} \qquad H(x,s)=\frac{3CL}{12\alpha^2+CL^3}xs\left(\frac{x}{L}-1\right)\left(\frac{s}{L}-1\right).$$

Therefore, the inverse operator F^{-1} defined by

$$F^{-1}[f](x)=\int_0^L (G+H)(x,s)f(s)ds,$$

$$F^{-1}: L^2[0,L]\to L^2[0,L],$$

is an integral operator with a continuous, symmetric kernal. Thus F^{-1} is compact and self-adjoint and we can say that the set of eigenfunctions for F^{-1} is a basis for $L^2[0,L]$. To see why this is exactly the set S defined above, we first consider the restriction

$$F_0^{-1}: H^2[0,L]\to L^2[0,L].$$

For any f in the domain of F_0^{-1}, f is continuous, and an application of F to both sides is permissible. Differentiation and integration show that

$$F[F_0^{-1}[f]]=f, \qquad \forall f\epsilon H^2[0,L].$$

Now for any X in the domain of F_0^{-1} such that

$$F_0^{-1}[X]=\Lambda X,$$

we have

$$F[F_0^{-1}[X]]=F[\Lambda X]=\Lambda F[X]=X$$

which implies that X is an eigenfunction for F associated with the eigenvalue $\frac{1}{\Lambda}$. Thus the eigenfunctions of F_0^{-1} are contained in S. Similarly, since $F_0^{-1}[F[X]]=X \quad \forall X\epsilon D(F)$, S is contained in the set of eigenfunctions of F_0^{-1}, and we have set equality. Finally, since $H^2[0,L]$ is dense in $L^2[0,L]$, F^{-1} is the closure of F_0^{-1}, and S is the set of eigenfunctions of F^{-1}. This completes the proof.

For the sake of illustration, the first five elements of S (for F with $\alpha^2=L=1$, and $C=4\pi^2$) are graphed in Figure 5.

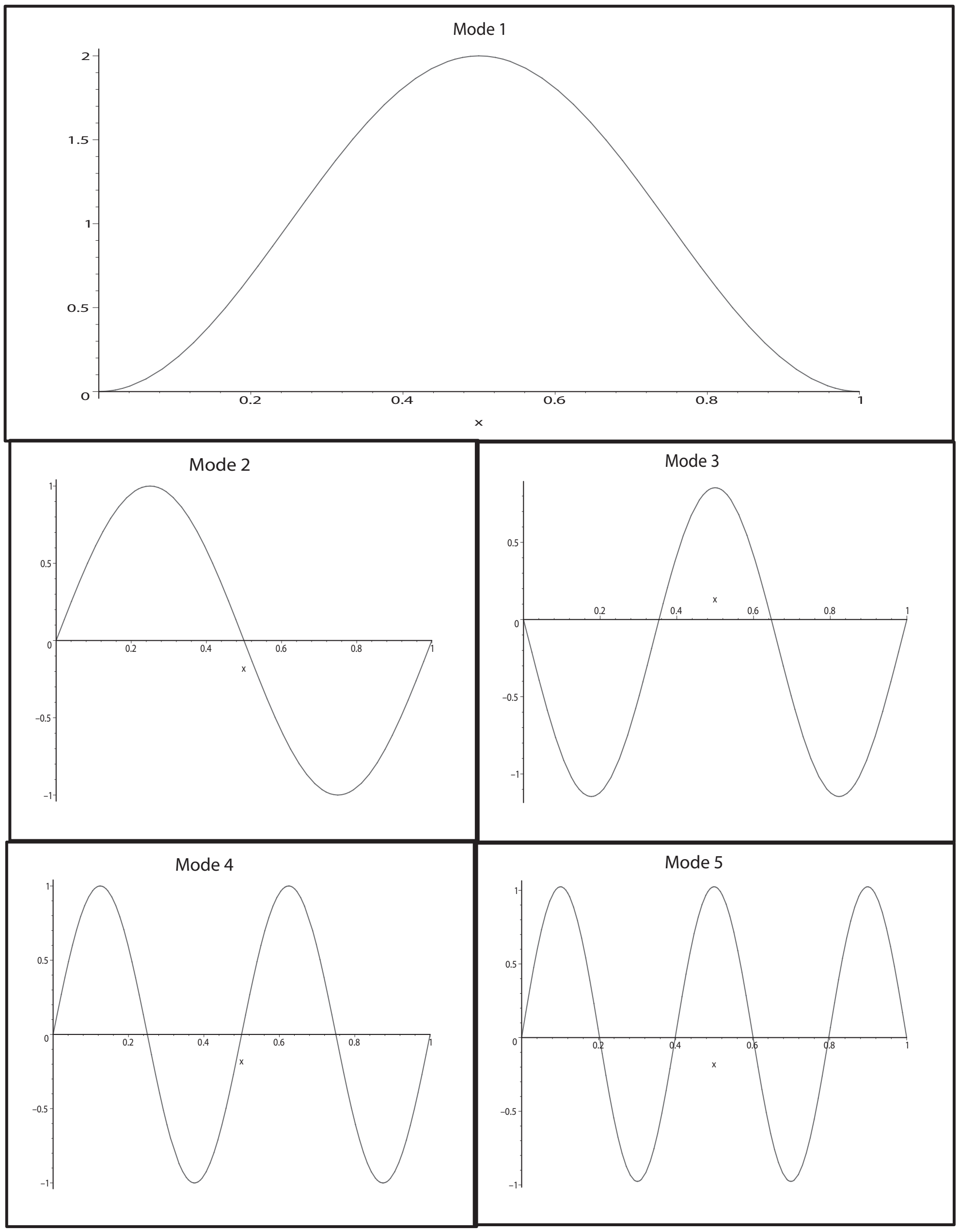

Figure 5: Shapes of the first five modes. Note that the odd modes are shifted sinusoids, and with the exception of the first one, they have non-integer frequencies.

Remark

In case (i) of the proof it is interesting to note that if for some positive integer m^* we have $CL^3 - 4(m^*)2\pi^2\alpha^2 = 0$, then there are no restrictions on ϕ and

$$X^*(x) = \sin\left(\frac{2m^*\pi x}{L} + \phi\right) - \sin(\phi)$$

is a solution to (4) for any choice of ϕ in the real numbers! To see why this is true, we first note that $m* = \frac{\sqrt{CL^3}}{2\alpha\pi}$ and that m^* is an *integer* solution of

$$\tan(\pi m) = \pi m - \frac{4\alpha^2\pi^3 m^3}{CL^3}.$$

This means that we have two different eigenfunctions associated with the eigenvalue $\Lambda^* = \frac{-4(m^*)^2\pi^2}{L^2}$, namely

$$S_{2m^*}(x) = \sin\left(\frac{2m^*\pi x}{L}\right), \qquad \textit{and} \qquad S_{2v_{m^*}}(x) = \cos\left(\frac{2m^*\pi x}{L} - m^*\pi\right) - \cos(m^*\pi).$$

Now

$$\begin{aligned}
X^*(x) &= \sin\left(\frac{2m^*\pi x}{L} + \phi\right) - \sin(\phi) \\
&= \sin\left(\frac{2m^*\pi x}{L}\right)\cos(\phi) + \cos\left(\frac{2m^*\pi x}{L}\right)\sin(\phi) - \sin(\phi) \\
&= \cos(\phi)\sin\left(\frac{2m^*\pi x}{L}\right) + \sin(\phi)\left[\cos\left(\frac{2m^*\pi x}{L}\right) - 1\right] \\
&= A\sin\left(\frac{2m^*\pi x}{L}\right) + B\left[\cos\left(\frac{2m^*\pi x}{L} - m^*\pi\right) - \cos(m^*\pi)\right]
\end{aligned}$$

where $A = \cos(\phi)$, and $B = \left(-1^{m^*}\right)\sin(\phi)$. Here we have taken into account the fact that m^* can be even or odd. Therefore

$$X^*(x) = AS_{2m^*}(x) + BS_{2v_{m^*}}(x)$$

and

$$\begin{aligned}
F\left[X^*\right](x) &= F\left[AS_{2m^*}\right](x) + F\left[BS_{2v_{m^*}}\right](x) \\
&= A\Lambda^* S_{2m^*}(x) + B\Lambda^* S_{2v_{m^*}}(x) \\
&= \Lambda^* X^*(x).
\end{aligned}$$

This phenomenon is reminiscent of the "degenerate modes" encountered in the solutions to the classical 2-D vibrating membrane with a square boundary, and it might offer an explanation for the so-called "sweet spot" or optimum tension which each drum and head combination seems to possess. Since adjusting the tension of the membrane would amount to changing the value of α, one could aim for a certain m^* if they desired a certain frequency to have the degenerate modes mentioned above. However, it is still not clear what frequency to strive for, and how or even if this phenomenon contributes to the resonance of a real drum.

Application

Using the values of Λ associated with each element of S, we can solve the O.D.E. (3) for the time dependent parts of each solution, and then write down the general solution to (1) using the principle of superposition for linear operators:

$$\begin{aligned} u_g(t,x) &= \sum_{n=1}^{\infty}\left[a_{2v_n}\cos\left(\frac{\alpha 2v_n\pi t}{L}\right)+b_{2v_n}\sin\left(\frac{\alpha 2v_n\pi t}{L}\right)\right]\left(\cos\left(\frac{2v_n\pi x}{L}-v_n\pi\right)-\cos(v_n\pi)\right) \\ &\quad+\sum_{m=1}^{\infty}\left[a_{2m}\cos\left(\frac{2m\alpha\pi t}{L}\right)+b_{2m}\sin\left(\frac{2m\alpha\pi t}{L}\right)\right]\sin\left(\frac{2m\pi x}{L}\right). \end{aligned}$$

Evaluating this expression and its first time derivative at $t=0$, we see that

$$u_g(0,x)=\sum_{n=1}^{\infty}a_{2v_n}\left(S_{2v_n}(x)\right)+\sum_{m=1}^{\infty}a_{2m}\left(S_{2m}(x)\right), \quad \textit{and}$$

$$\frac{\partial u_g(0,x)}{\partial t}=\sum_{n=1}^{\infty}\frac{2v_n\alpha\pi}{L}b_{2v_n}\left(S_{2v_n}(x)\right)+\sum_{m=1}^{\infty}\frac{2m\alpha\pi}{L}b_{2m}\left(S_{2m}(x)\right).$$

This, combined with the previous result allows us to solve initial value problems of the type

$$\frac{\partial^2 u_p(t,x)}{\partial t^2}=\alpha^2\frac{\partial^2 u_p(t,x)}{\partial x^2}-C\int_0^L u_p(t,x)dx,$$

$$u_p(t,0)=0=u_p(t,L) \quad \forall t>0,$$

$$u_p(0,x)=f(x), \qquad \frac{\partial u_p(0,x)}{\partial t}=g(x)$$

because as long as f is continuous and g is piecewise continuous, we may find the eigenfunction expansions

$$f(x)=\sum_{n=1}^{\infty}s^f_{2v_n}S_{2v_n}(x)+\sum_{m=1}^{\infty}s^f_{2m}S_{2m}(x),$$

$$g(x)=\sum_{n=1}^{\infty}s^g_{2v_n}S_{2v_n}(x)+\sum_{m=1}^{\infty}s^g_{2m}S_{2m}(x),$$

and construct the particular solution

$$\begin{aligned} u_p(t,x) &= \sum_{n=1}^{\infty}\left[s^f_{2v_n}\cos\left(\frac{2v_n\alpha\pi t}{L}\right)+\frac{L}{2v_n\alpha\pi}s^g_{2v_n}\sin\left(\frac{2v_n\alpha\pi t}{L}\right)\right]S_{2v_n}(x) \\ &\quad+\sum_{m=1}^{\infty}\left[s^f_{2m}\cos\left(\frac{2m\alpha\pi t}{L}\right)+\frac{L}{2m\alpha\pi}s^g_{2m}\sin\left(\frac{2m\alpha\pi t}{L}\right)\right]S_{2m}(x). \end{aligned}$$

Coupled Membranes

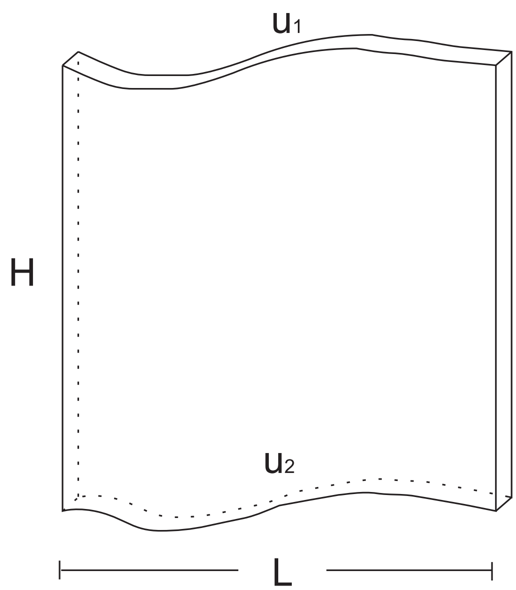

Figure 6: Geometry of the idealized drum with two membranes.

Here again we consider a simplified model of a drum with two flexible membranes of length L, rigid sides with height H, and a width of 1 unit. Using the same assumptions as before, the following system of integro-differential equations can be derived:

$$\text{(7)} \qquad \frac{\partial^2 u_1(t,x)}{\partial t^2} = \alpha_1^2 \frac{\partial^2 u_1(t,x)}{\partial x^2} - C_1 \int_0^L [u_1 - u_2](t,x)dx,$$

$$\text{(8)} \qquad \frac{\partial^2 u_2(t,x)}{\partial t^2} = \alpha_2^2 \frac{\partial^2 u_2(t,x)}{\partial x^2} + C_2 \int_0^L [u_1 - u_2](t,x)dx.$$

Solutions

With the example of a bass drum in mind, we take $\alpha_1 = \alpha_2 = \alpha$, and $C_1 = C_2 = C$ This represents the idealized scenario in which both drumheads have the same density and are tuned to the same tension. With this setup, we look for pairs of solutions $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1 = -u_2$. These represent the modes of vibration where the heads move opposite each other, alternately compressing and expanding the enclosed volume of air. Thus, (7) and (8) become

$$\text{(9)} \qquad \frac{\partial^2 u(t,x)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(t,x)}{\partial x^2} - 2C \int_0^L u(t,x)dx,$$

where $u = u_1 = -u_2$. Using the previous result for the one membrane problem it can be shown that

$$u_n(t,x) = \left[a_n \sin\left(\frac{2v_n\pi\alpha t}{L}\right) + b_n \cos\left(\frac{2v_n\pi\alpha t}{L}\right)\right] S_{2v_n}(x), \qquad a_n, b_n, \epsilon\mathbf{R}$$

is a solution to (9) where S is defined as above, with v_n solving

$$\tan(\pi v) = \pi v - \frac{4\alpha^2\pi^3 v^3}{2CL^3} \qquad for \quad n = 1, 2, \ldots.$$

Furthermore, if we look for pairs of solutions to (7)-(8) where $u_1 = u_2 = u$, and $\int_0^L u(t,x)dx \neq 0$, then (7) and (8) become

$$(10) \qquad \frac{\partial^2 u(t,x)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad \textit{and}$$

$$u_m(t,x) = \left[c_m \sin\left(\frac{(2m-1)\pi\alpha t}{L}\right) + d_m \sin\left(\frac{(2m-1)\pi\alpha t}{L}\right)\right] \sin\left(\frac{(2m-1)\pi x}{L}\right), \qquad c_m, d_m \epsilon \mathbf{R}$$

is a solution to (10) for $m = 1, 2, \ldots$. These are the "odd" solutions to the vibrating string problem, and they represent the modes of vibration where both heads move "in sync" with each other. Finally, if $\int_0^L u_1(t,x)dx = 0 = \int_0^L u_2(t,x)dx$, then (7) and (8) become uncoupled, and we arrive at the "even" solutions

$$\begin{bmatrix} u_1(t,x) \\ u_2(t,x) \end{bmatrix} = \begin{bmatrix} \left(p_k \sin\left(\frac{2k\pi\alpha t}{L}\right) + q_k \sin\left(\frac{2k\pi\alpha t}{L}\right)\right) \sin\left(\frac{2k\pi x}{L}\right) \\ \left(r_k \sin\left(\frac{2k\pi\alpha t}{L}\right) + s_k \sin\left(\frac{2k\pi\alpha t}{L}\right) \sin\left(\frac{2k\pi x}{L}\right)\right) \end{bmatrix}$$

$$p_k, q_k, r_k, s_k \epsilon \mathbf{R}, \qquad k = 1, 2, \ldots.$$

Now we can write down the general solution to (7)-(8) as follows:

$$(11) \qquad \begin{bmatrix} u_1(t,x) \\ u_2(t,x) \end{bmatrix}_g = \underbrace{\sum_{n=1}^{\infty} \left(a_n \begin{bmatrix} \sin\left(\frac{2v_n\pi\alpha t}{L}\right) \\ -\sin\left(\frac{2v_n\pi\alpha t}{L}\right) \end{bmatrix} + b_n \begin{bmatrix} \cos\left(\frac{2v_n\pi\alpha t}{L}\right) \\ \cos\left(\frac{2v_n\pi\alpha t}{L}\right) \end{bmatrix} \right) S_{2v_n}(x)}_{v}$$

$$+ \underbrace{\sum_{m=1}^{\infty} \left(c_m \begin{bmatrix} \sin\left(\frac{(2m-1)\pi\alpha t}{L}\right) \\ \sin\left(\frac{(2m-1)\pi\alpha t}{L}\right) \end{bmatrix} + d_m \begin{bmatrix} \cos\left(\frac{(2m-1)\pi\alpha t}{L}\right) \\ \cos\left(\frac{(2m-1)\pi\alpha t}{L}\right) \end{bmatrix} \right) \sin\left(\frac{(2m-1)\pi x}{L}\right)}_{w}$$

$$+ \underbrace{\sum_{k=1}^{\infty} \begin{bmatrix} p_k \sin\left(\frac{2k\pi\alpha t}{L}\right) + q_k \cos\left(\frac{2k\pi\alpha t}{L}\right) \\ r_k \sin\left(\frac{2k\pi\alpha t}{L}\right) + s_k \cos\left(\frac{2k\pi\alpha t}{L}\right) \end{bmatrix} \sin\left(\frac{2k\pi x}{L}\right)}_{y}$$

where v represents the "out-of-phase" solutions, w represents the "in-phase" solutions, and y represents the uncoupled solutions. It is worthwhile to note that since $\int_0^L y(t,x)dx = 0$, any fluctuations in volume must come from the u and v solutions.

The claim here is that for any initial displacements and velocities

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \quad \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}, \qquad f_1, f_2, g_1, g_2 \epsilon L^2[0, L]$$

the constants in the above solution can be chosen so that

$$\begin{bmatrix} u_1(0,x) \\ u_2(0,x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \qquad \textit{and} \qquad \begin{bmatrix} \frac{\partial u_1(0,x)}{\partial t} \\ \frac{\partial u_2(0,x)}{\partial t} \end{bmatrix} = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}.$$

To see why this is true, we first evaluate the general solution and its time derivative at $t = 0$, obtaining

$$\left[\begin{array}{c} u_1(0,x) \\ u_2(0,x) \end{array} \right] = \sum_{n=1}^{\infty} b_n S_{2v_n}(x) \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + \sum_{m=1}^{\infty} d_m \sin\left(\frac{(2m-1)\pi x}{L}\right) \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$
$$+ \sum_{k=1}^{\infty} \left[\begin{array}{c} q_k \\ s_k \end{array} \right] \sin\left(\frac{2k\pi x}{L}\right),$$

and $$\left[\begin{array}{c} \frac{\partial u_1(0,x)}{\partial t} \\ \frac{\partial u_2(0,x)}{\partial t} \end{array} \right] = \sum_{n=1}^{\infty} 2v_n \pi \alpha a_n S_{2v_n} \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + \sum_{m=1}^{\infty} (2m-1)\pi\alpha c_m \sin\left(\frac{(2m-1)\pi x}{L}\right) \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$
$$+ \sum_{k=1}^{\infty} 2k\pi\alpha \left[\begin{array}{c} p_k \\ r_k \end{array} \right] \sin\left(\frac{2k\pi x}{L}\right).$$

In order to decompose the initial displacements in terms of the first expression, we start by finding the "even" Fourier coefficients of $f_1(x)$ and $f_2(x)$. Then we can write

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right] = \left[\begin{array}{c} \hat{f}_1(x) + \bar{f}_1(x) \\ \hat{f}_2(x) + \bar{f}_2(x) \end{array} \right],$$

where $$\left[\begin{array}{c} \bar{f}_1(x) \\ \bar{f}_2(x) \end{array} \right] = \sum_{k=1}^{\infty} \left[\begin{array}{c} q_k \\ s_k \end{array} \right] \sin\left(\frac{2k\pi x}{L}\right).$$

The residual $\left[\begin{array}{c} \hat{f}_1(x) \\ \hat{f}_2(x) \end{array} \right] = \left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right] - \left[\begin{array}{c} \bar{f}_1(x) \\ \bar{f}_2(x) \end{array} \right]$ can be written as

$$\left[\begin{array}{c} \hat{f}_1(x) \\ \hat{f}_2(x) \end{array} \right] = v_f(x) \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + w_f(x) \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$

where $$v_f(x) = \frac{\hat{f}_1(x) - \hat{f}_2(x)}{2}, \qquad \textit{and} \quad w_f(x) = \frac{\hat{f}_1(x) + \hat{f}_2(x)}{2}.$$

Therefore we need only to express $v_f(x)$ in terms of the elements of S, and $w_f(x)$ in terms of the "odd" sine functions. From the previous result on the completeness and orthogonality of S, it is clear that this can be done. Thus, if $v_f(x) = \sum_{n=1}^{\infty} b_n S_{2v_n}(x)$ and $w_f(x) = \sum_{m=1}^{\infty} d_m \sin\left(\frac{(2m-1)\pi x}{L}\right)$, then

$$\begin{aligned} \left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right] &= \left[\begin{array}{c} \hat{f}_1(x) \\ \hat{f}_2(x) \end{array} \right] + \left[\begin{array}{c} \bar{f}_1(x) \\ \bar{f}_2(x) \end{array} \right] \\ &= \sum_{n=1}^{\infty} b_n S_{2v_n}(x) \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + \sum_{m=1}^{\infty} d_m \sin\left(\frac{(2m-1)\pi x}{L}\right) \left[\begin{array}{c} 1 \\ 1 \end{array} \right] + \sum_{k=1}^{\infty} \left[\begin{array}{c} q_k \\ s_k \end{array} \right] \sin\left(\frac{2k\pi x}{L}\right). \end{aligned}$$

This means that the general solution (11), with the appropriate $b_n's$, $d_m's$, $q_k's$, and $s_k's$, can be made to fit any initial displacements. A similar process can be used to fit the initial velocities.

Application

In order to model a struck bass drum with identical heads and tensions, the initial value problem (9) − (10) $(with\ \alpha = 2,\ C = \pi^2,\ and\ L = 1)$ was solved with the following combination of initial displacements and velocities.

$$\begin{bmatrix} u_1(0,x) \\ u_2(0,x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial u_1(0,x)}{\partial t} \\ \frac{\partial u_2(0,x)}{\partial t} \end{bmatrix} = \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix}$$

$$where \qquad g_1(x) = \begin{cases} 0, & 0 \le x < 0.4, \\ -1, & 0.4 \le x \le 0.6, \\ 0, & 0.6 < x \le 1. \end{cases}$$

The first twenty terms in each series were computed by using approximations to the first twenty functions in S. The resulting pair of approximate solutions describes the upper and lower membranes' displacements over time and is shown in Figure 8. In an effort to simulate the proximity sensor data obtained by Obata and Tesima, the displacement of the fixed point $x = \frac{1}{4}$ on each membrane was graphed over the first 5.5 seconds, and the plots are shown in Figure 7.

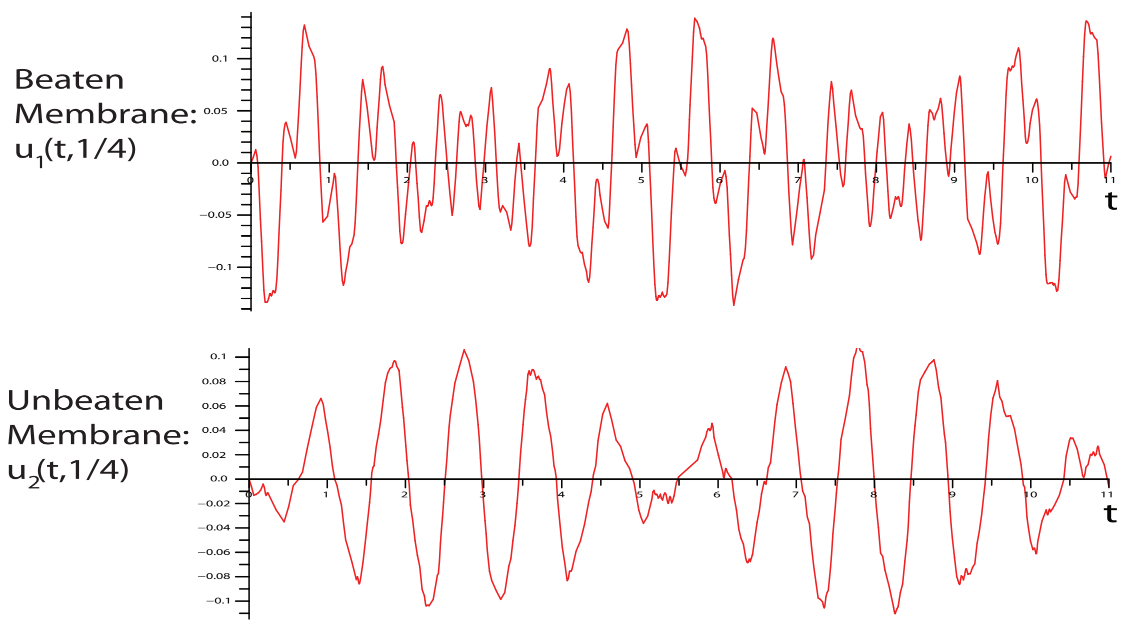

Figure 7: Tracking the displacement of the point $x = \frac{1}{4}$ on the top and bottom membranes over time.

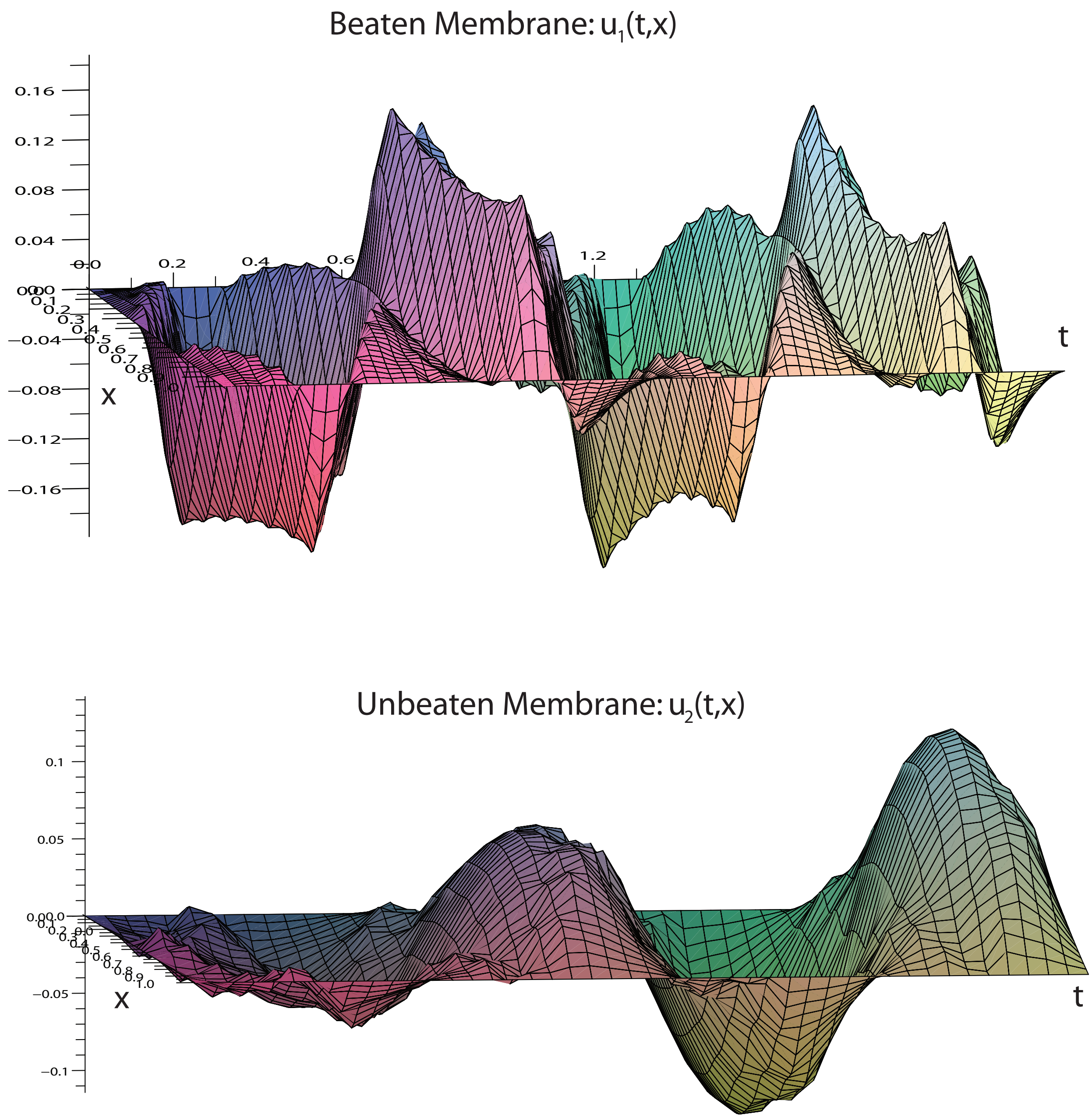

Figure 8: Displacements of the top and bottom membranes in a coupled system over time.

Extension

When both membranes have the same characteristics, the above solution technique is relatively straightforward, although a bit labor-intensive. A considerably more challenging problem is posed by the possibility of using different materials for the top and bottom heads of the drum, or even just tuning them differently. This amounts to solving (7)-(8) with $\alpha_1 \neq \alpha_2$ or $C_1 \neq C_2$.

Once this problem is solved, the techniques presented in this paper could be extended to allow displacements in the y-direction as well, and models for three-dimensional box-shaped drums could be developed. By writing the integro-differential equations in polar coordinates, it would be possible to develop a mathematical model for cylindrical drums, although the solutions may be quite a bit harder to come by.

Finally, the phenomenon of "movable nodes" that was remarked upon earlier could be investigated further with the intent of predicting the "fundamental tone" of a given drum and head combination.

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